Theorem (Gauss). *Let $p \neq q \in \mathbb{Z}^+$ be odd primes. Then:* $\left(\frac{p}{q}\right) \cdot \left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}$.

Proof. Let $G(n) := \sum_{x \in \mathbb{Z}/n} e^{2\pi i x^2/n}$, and note that, since $G(pq) = \left(\frac{p}{q}\right) \cdot \left(\frac{q}{p}\right) \cdot G(p) \cdot G(q)$, it suffices to show that $G(n) = \sqrt{n} \cdot \begin{cases} 1 & n \equiv 1 \pmod 4 \\ i & n \equiv 3 \pmod 4 \end{cases}$ for odd $n \in \mathbb{Z}^+$ with e.g. at most two prime factors (evidently $|G(n)|^2 = n$). Let $\varphi \in C^\infty\left(\left[-\frac{n}{4}, \frac{3n}{4}\right]\right)$ be such that $\varphi|_{\left[-\frac{n+1}{4}, \frac{3n-1}{4}\right]} = 1$, so that

$$G(n) = \sum_{x \in \mathbb{Z}} \varphi(x) \cdot e^{2\pi i x^2/n} = \sum_{\xi \in \mathbb{Z}} \int_{\mathbb{R}} dx\, \varphi(x) \cdot e^{2\pi i \left(\frac{x^2}{n} - x \cdot \xi\right)} = n \cdot \sum_{\xi \in \mathbb{Z}} \int_{-\frac{1}{4}}^{\frac{3}{4}} dx\, \varphi(n \cdot x) \cdot e^{2\pi i n \cdot \left(x^2 - x \cdot \xi\right)}.$$

After repeated use of the integration by parts identity (note also that $\varphi'|_{\left[-\frac{n+1}{4}, \frac{3n}{4}\right]} = 0$, so that $\varphi'(n \cdot x)$ is only nonzero on an interval of length $\ll \frac{1}{n}$) $\int f \cdot e^g = -\int \left(\frac{f}{g'}\right)' \cdot e^g$, we see that the contribution of the terms with $\xi \neq 0, 1$ is $\ll 1$. Thus

$$G(n) = O(1) + \int_{-\frac{n}{4}}^{\frac{3n}{4}} dx\, \varphi(x) \cdot e^{2\pi i x^2/n} + \int_{-\frac{n}{4}}^{\frac{3n}{4}} dx\, \varphi(x) \cdot e^{2\pi i \left(x^2/n - x\right)} = O(1) + (1 + i^{-n}) \cdot \sqrt{n} \cdot \int_{-\frac{\sqrt{n}}{4}}^{\frac{3\sqrt{n}}{4}} dx\, e^{2\pi i x^2}.$$

Because $\int_{\mathbb{R}} dx\, e^{2\pi i x^2} = \frac{1+i}{2}$, we see that $G(n) = \sqrt{n} \cdot \left(\frac{(1+i^{-n}) \cdot (1+i)}{2} + o(1)\right)$. Replacing n by n^{2k+1} via $G(n^{2k+1}) = n^k \cdot G(n)$ and taking $k \to \infty$ we deduce $G(n) = \sqrt{n} \cdot \frac{(1+i^{-n}) \cdot (1+i)}{2}$ as desired.